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RIGID FAMILIES AND ENDOMORPHISM ALGEBRAS OF KRONECKER MODULES

BY

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ABSTRACT

Let R be a commutative ring with an identity element and let $\Gamma_2(R) = \begin{pmatrix} R & R^2 \\ 0 & R \end{pmatrix}$ be the the Kronecker R-algebra. One of our main results is Theorem 1.2 asserting that for any R-algebra A generated by λ elements, where λ is an infinite cardinal number, there exists a rigid direct system $\mathbb{F} = \{\mathbb{F}_\beta, f_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ (see Definition 2.8) of fully faithful R-linear exact functors $\mathbb{F}_\beta\colon \mathrm{Mod}(A)\longrightarrow\mathrm{Mod}(\Gamma_2(R))$ connected by R-splitting functorial monomorphisms $f_{\beta\gamma}\colon \mathbb{F}_\beta \to \mathbb{F}_\gamma$ satisfying some extra conditions. In particular, if R is a field then every R-algebra generated by at most λ elements is isomorphic to an endomorphism algebra End X of a Kronecker module $X = (X', X'', \varphi', \varphi'')$ in $\mathrm{Mod}\Gamma_2(R)$ such that $\dim_R X' = \dim_R X'' = \lambda$, the R-linear maps $\varphi', \varphi''\colon X' \to X''$ are injective and $X'' = \mathrm{Im}\ \varphi' + \mathrm{Im}\ \varphi''$.

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1. Introduction

Throughout this paper K is a field and R is a commutative ring with an identity element.

Following Corner [4] many authors have studied nice subcategories $\mathcal{A}$ of a module category $\mathrm{Mod}(\Lambda)$ over finitely generated R-algebras Λ for which there exist large objects in $\mathcal{A}$ with prescribed endomorphism R-algebras and rigid direct systems of objects in $\mathcal{A}$ (see [1], [2], [6], [8], [9], [10], [14], [15]). The existence problem reduces to a corresponding problem for the category $\mathrm{Mod}(\Gamma_2(R))$ of Kronecker R-modules defined below, if there exists a full faithful and exact functor $T : \mathrm{Mod}(\Gamma_2(R)) \longrightarrow \mathcal{A}$. Fortunately, such a functorial embedding can be constructed for many interesting subcategories $\mathcal{A}$ (see [11], [17], [18], [21]).

In the present paper we solve the existence problem in the affirmative for the category $\mathrm{Mod}(\Gamma_2(R))$ of Kronecker R-modules (see Theorem 1.2 and Corollary 1.3). Our main result is successfully applied in [11], where among other things we get an alternative and short proof of Theorem 2 of [10].

We recall that a right module X over a generalized triangular matrix ring

$$S = \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}$$

can be identified with the system

$$X = (X'_A, X''_B, \varphi),$$

where X'_A is a right A-module, X''_B is a right B-module and φ: $X' \otimes_A M_B \to X''_B$ is a B-homomorphism (see [16]). We recall from [19] that the S-module X is said to be **propartite** if X'_A is a projective A-module and X''_B is a projective B-module. We denote by $\mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(S)^A_B$ the category of propartite right S-modules, and by $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(S)^A_B$ the full subcategory of $\mathrm{Mod}^{\mathrm{pr}}_{\mathrm{pr}}(S)^A_B$ consisting of finitely generated modules.

For any ring A with an identity element, the generalized matrix A-algebra

$$\Gamma_2(A) = \begin{pmatrix} A & A^2 \\ 0 & A \end{pmatrix} \tag{1.1}$$

is called the **Kronecker A-algebra**, where the multiplication is defined naturally by the formula

$$\begin{pmatrix} d & u \\ 0 & c \end{pmatrix} \begin{pmatrix} f & v \\ 0 & e \end{pmatrix} = \begin{pmatrix} df & dv + ue \\ 0 & ce \end{pmatrix}.$$

The right $\Gamma_2(A)$-modules are called **Kronecker A-modules.** Following the convention introduced above the category $\mathrm{Mod}(\Gamma_2(A))$ of Kronecker A-modules

X can be identified with the category of A-representations of the Kronecker quiver (see [14] and [16])

$$\bullet \overset{\varphi'}{\underset{\varphi''}{\rightrightarrows}} \bullet$$

that is, the systems

$$X = (X', X'', \varphi', \varphi'')$$

where X' and X'' are A-modules and $\varphi', \varphi''\colon X' \to X''$ are A-homomorphisms. A morphism from $X = (X', X'', \varphi', \varphi'')$ to $X_1 = (X_1', X_1'', \varphi_1', \varphi_1'')$ is a pair (f', f'') of A-module homomorphisms $f' : X' \to X_1'$, $f'' : X'' \to X_1''$ such that $\varphi_1' f' = f''\varphi'$ and $\varphi_1'' f' = f''\varphi''$.

The category $\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(\Gamma_2(A))$ of propartite $\Gamma_2(A)$-modules will be called the category of **A-projective Kronecker modules**. It is easy to see that $\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(\Gamma_2(A))$ can be identified with the category of A-projective representations of the Kronecker quiver, that is, the A-representations $P = (P', P'', \varphi', \varphi'')$, where P' and P'' are projective A-modules.

The following theorem is the main result of this paper.

THEOREM 1.2: *Let R be a commutative ring with an identity element and let*

$$\Gamma_2(R) = \begin{pmatrix} R & R^2 \\ 0 & R \end{pmatrix}$$

be the Kronecker R-algebra (1.1).

(a) *For any R-algebra A generated by λ elements, where λ is an infinite cardinal number, there exists a direct system*

$$\mathbb{F} = \{\mathbb{F}_\beta, f_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$$

of full faithful R-linear exact functors $\mathbb{F}_\beta\colon \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(\Gamma_2(R))$ connected by injective functorial morphisms $f_{\beta\gamma}\colon \mathbb{F}_\beta \to \mathbb{F}_\gamma$ satisfying the following conditions:

(i) *For every A-free module M in $\mathrm{Mod}(A)$, the Kronecker R-module $\mathbb{F}_\beta(M) = (M_\beta', M_\beta'', \varphi_\beta', \varphi_\beta'')$ is A-free for all $\beta \subseteq \lambda$, the free A-modules M_β' and M_β'' are of rank λ, $\varphi_\beta', \varphi_\beta''\colon M_\beta' \longrightarrow M_\beta''$ are A-module monomorphisms such that $\mathrm{Im}\,\varphi_\beta' + \mathrm{Im}\,\varphi_\beta'' = M_\beta''$ and the modules $M_\beta'/\mathrm{Im}\,\varphi'$, $M_\beta''/\mathrm{Im}\,\varphi''$ are A-free. In particular, if M is A-free, the Kronecker R-module $\mathbb{F}_\beta(M)$ lies in the category $\mathrm{Mod}_{\mathrm{pr}}^{\mathrm{pr}}(\Gamma_2(A))$ and has no direct summands of the form $X = (X', X'', \varphi', \varphi'')$, where $X' = 0$ and X'' is a projective A-module.*

(ii) *For every module M in $\mathrm{Mod}(A)$ and for all $\beta \subseteq \gamma \subseteq \lambda$ the $\Gamma_2(R)$-homomorphism $f_{\beta\gamma}(M) : \mathbb{F}_\beta(M) \to \mathbb{F}_\gamma(M)$ is an R-splittable monomorphism.*

(iii) *If M and N are modules in* $\mathrm{Mod}(A)$ *then*

$$\mathrm{Hom}_{\Gamma_2(R)}(\mathbb{F}_\beta(M), \mathbb{F}_\gamma(N)) = 0 \quad \textit{if } \beta \not\subseteq \gamma,$$

and the natural R-homomorphism

$$\mathrm{Hom}_A(M,N) \xrightarrow{\ \simeq\ } \mathrm{Hom}_{\Gamma_2(R)}(\mathbb{F}_\beta(M), \mathbb{F}_\gamma(N)), \quad g \mapsto f_{\beta\gamma}(N) \circ \mathbb{F}_\beta(g),$$

is an isomorphism for all $\beta \subseteq \gamma \subseteq \lambda$.

(b) *Any R-algebra A is isomorphic to a $\Gamma_2(R)$-endomorphism algebra of the form* End X, *where X is an A-free Kronecker A-module in* $\mathrm{Mod}(\Gamma_2(R))$.

An immediate consequence of Theorem 1.2 is the following generalization of a well-known result of Ringel [14, Corollary, p. 407].

COROLLARY 1.3: *Let K be an arbitrary field. Every K-algebra generated by at most λ elements, where λ is an infinite cardinal number, is isomorphic to an endomorphism algebra* End X *of a Kronecker module $X = (X', X'', \varphi', \varphi'')$ in* $\mathrm{Mod}\begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix}$ *such that $\dim_K X' = \dim_K X'' = \lambda$, the K-linear maps φ' and φ'' are injective and $X'' = \mathrm{Im}\ \varphi' + \mathrm{Im}\ \varphi''$.*

Note that in case R is a field K our Theorem 1.2 and Corollary 1.3 are close to the fact proved by Ringel [14, Corollary, p. 408] and asserting that every hereditary representation-infinite K-algebra Λ of finite dimension is "WILD" in the sense defined in [14, p. 408]. In particular, the result of Ringel [14, p. 408] provides us with a full and faithful embedding functor

$$\mathbb{F}\colon \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}\begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix}$$

for any K-algebra A.

Let us remark that our Theorem 1.2 is more general than the result of Ringel, because it implies the existence of such an embedding $\mathbb{F}$ satisfying in addition the conditions stated in (i) of Theorem 1.2. Moreover, it guarantees the existence of a rigid family of functors $\mathbb{F}_\lambda$ satisfying the conditions (i)–(iii) of Theorem 1.2.

The organization of the paper is as follows. In Section 2 we collect basic definitions and facts on λ-families and rigid systems. We mainly follow the notations and terminology introduced in [9] and [10].

The proof of Theorem 1.2 is presented at the end of Section 3. It depends on several preparatory results proved in Section 3; the main ones are Theorem 3.3 and Proposition 3.5.

Theorem 1.2 is valid for any infinite cardinals λ; however, for sake of clarity we restrict our consideration to regular cardinals. The passage to singular cardinals does not depend on the particular modules and is given in [9].

An application of Theorem 1.2 is given in Corollary 4.5. Some open problems are discussed in Section 4.

2. Generalities on λ-families and rigid systems

Assume that R is a commutative ring with an identity element and $\mathcal{A}$ is a full subcategory of the category $\mathrm{Mod}(\Lambda)$, where Λ is an R-algebra. Throughout we usually assume that the subcategory $\mathcal{A}$ is closed under taking arbitrary direct sums and under taking extensions in $\mathrm{Mod}(\Lambda)$.

A sequence $0 \to Y' \to X \to Y'' \to 0$ in $\mathcal{A}$ is said to be exact in $\mathcal{A}$ if it is an exact sequence in $\mathrm{Mod}(\Lambda)$.

We start this section by recalling from [9] and [10] the definitions and basic facts on λ-families.

Let λ be an infinite cardinal number and let $\mathcal{A}$ be an arbitrary full subcategory of the category $\mathrm{Mod}(\Lambda)$, where Λ is an R-algebra. A **λ-family** in $\mathcal{A}$ is the λ-directed system

$$\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda} \tag{2.1}$$

in $\mathcal{A}$, that is, U_β is an object in $\mathcal{A}$ and $u_{\beta\gamma}\colon U_\beta \to U_\gamma$ is a Λ-homomorphism in $\mathcal{A}$ for $\beta \subseteq \gamma$, such that $u_{\beta\beta}$ is the identity map on U_β, and if $\alpha \subseteq \beta \subseteq \gamma$ then $u_{\alpha\gamma} = u_{\beta\gamma} \circ u_{\alpha\beta}$.

We say that the λ-family $\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ has the **R-mono-splitting property (resp. is R-free)** if for all $\beta \subseteq \gamma \subseteq \lambda$ the homomorphism $u_{\beta\gamma} : U_\beta \to U_\gamma$ is an R-splittable monomorphism (resp. $u_{\beta\gamma}$ is injective and the modules U_β, U_γ, $\mathrm{Coker}\, u_{\beta\gamma}$ are R-free). If in addition the free R-modules U_β, U_γ, $\mathrm{Coker}\, u_{\beta\gamma}$ are of rank λ, the λ-family is called **λ-R-free**.

A **weak λ-family** in $\mathcal{A}$ is a λ-family $\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ in $\mathcal{A}$ such that for all $\beta \cup \gamma \subseteq \lambda$ the sequence

$$0 \longrightarrow U_{\beta\cap\gamma} \longrightarrow U_\beta \oplus U_\gamma \longrightarrow U_{\beta\cup\gamma} \longrightarrow 0$$

is exact, where the maps are the natural ones induced by $u_{\beta\cap\gamma,\beta}$, $u_{\beta\cap\gamma,\gamma}$, $u_{\beta,\beta\cup\gamma}$ and $u_{\gamma,\beta\cup\gamma}$.

If M and N are R-modules, we say that the weak λ-family $\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ in $\mathcal{A}$ is **rigid for the pair M, N** if the following conditions hold:

(W1) *Direct sums of families for various $\alpha, \beta \subseteq \lambda$ of the induced homomorphism $N \otimes_R U_{\alpha\cap\beta} \to N \otimes_R (U_\alpha \oplus U_\beta)$ are semi-stable kernels (see [13]).*

(W2) *Any Λ-homomorphism $U_\alpha \to \bigoplus_{\beta\subseteq\lambda}(N \otimes_R U_\beta)$ factors through a finite direct sum.*

(W3) *If ψ: $U_\beta \to N \otimes_R U_\beta$ is a Λ-homomorphism such that the composed homomorphism $U_\emptyset \xrightarrow{u_{\emptyset\beta}} U_\beta \xrightarrow{\psi} N \otimes_R U_\beta$ is zero, then $\psi = 0$.*

(W4) $\mathrm{Hom}_\Lambda(M \otimes_R U_\beta, N \otimes_R U_\gamma) = \begin{cases} \mathrm{Hom}_R(M,N) & \text{if } \beta = \gamma, \\ 0 & \text{if } \beta \setminus \gamma \text{ is infinite,} \end{cases}$
where the equality "=" means that the canonical R-homomorphism

$$\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_\Lambda(M \otimes_R H_\beta, N \otimes_R H_\gamma), \tag{2.2}$$

$f \mapsto f \otimes u_{\beta\gamma}$, is bijective. If the family is rigid for every pair M, N of R-modules, we call it a **rigid family**.

Remarks 2.3: (a) Note that according to the tensor product adjoint formula the condition **(W4)** holds for every R-module M if and only if **(W4)** holds for $M = R$.

(b) If we assume that the full subcategory $\mathcal{A}$ of $\mathrm{Mod}(\Lambda)$ is closed under taking arbitrary direct sums and under taking extensions in $\mathrm{Mod}(\Lambda)$, then every weak λ-family $\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ in $\mathcal{A}$ has the property **(W1)**.

A **strong λ-family** in $\mathcal{A}$ is a system $\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$, where $\{H_\beta, h_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ is a weak λ-family in $\mathcal{A}$, and $\{Q_i\}_{i\in\lambda}$ is a family of objects in $\mathcal{A}$ together with homomorphisms $H_\beta \to \bigoplus_{i\in\beta} Q_i$ such that, whenever $\beta \subseteq \gamma$, there is a commutative diagram

$$\begin{array}{ccccccc} H_\emptyset & \longrightarrow & H_\beta & \longrightarrow & \bigoplus_{i\in\beta} Q_i & \longrightarrow & 0 \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h_{\beta\gamma}} & & \downarrow & & \\ H_\emptyset & \longrightarrow & H_\gamma & \longrightarrow & \bigoplus_{i\in\gamma} Q_i & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$ with exact rows, where the right-hand vertical map is the natural direct sum embedding.

It follows that the homomorphism $h_{\beta\gamma}$ is injective and there is an exact sequence

$$0 \longrightarrow H_\beta \xrightarrow{h_{\beta\gamma}} H_\gamma \longrightarrow \bigoplus_{i\in\gamma\setminus\beta} Q_i \longrightarrow 0$$

in $\mathcal{A}$ for all $\beta \subseteq \gamma \subseteq \lambda$.

If M and N are R-modules, we say that the strong λ-family $\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ in $\mathcal{A}$ is **strongly rigid for the pair M, N** if the following holds:

(S1) *The R-homomorphism $id_N \otimes h_{\beta\gamma}: N \otimes_R H_\beta \to N \otimes_R H_\gamma$ is injective for all $\beta \subseteq \gamma$.*

(S2) *The R-homomorphism*

$$\Psi_{N,Q}: N \longrightarrow \mathrm{Hom}_\Lambda(Q_i, N \otimes_R Q_i),$$

$n \mapsto (q \mapsto n \otimes q)$, is bijective for any $i \in \lambda$.

(S3) $\mathrm{Hom}_\Lambda(M \otimes_R H_\beta, N \otimes_R H_\gamma) = \begin{cases} \mathrm{Hom}_R(M,N) & \text{if } \beta \subseteq \gamma, \\ 0 & \text{if } \beta \not\subseteq \gamma, \end{cases}$

where the equality "=" means that the canonical R-homomorphism

$$\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_\Lambda(M \otimes_R H_\beta, N \otimes_R H_\gamma), \tag{2.4}$$

$f \mapsto f \otimes h_{\beta\gamma}$, is bijective.

If the family $\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ is strongly rigid for every pair M, N of R-modules, we call it a **strongly rigid family**.

Remark 2.5: According to the tensor product adjoint formula, the condition **(S3)** holds for every R-module M if and only if **(S3)** holds for $M = R$.

Throughout, we assume that R is a commutative ring with an identity element, Λ is an R-algebra and $\mathcal{A}$ is a full subcategory of the category $\mathrm{Mod}(\Lambda)$ being closed under taking arbitrary direct sums and under taking extensions in $\mathrm{Mod}(\Lambda)$.

The following proposition provides us with a useful reduction tool.

PROPOSITION 2.6: *Let λ be an infinite cardinal number. Assume that R, Λ and $\mathcal{A}$ are as above.*

(a) *For any weak λ-family $\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ in the category $\mathcal{A}$ there exists a strong λ-family $\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ in $\mathcal{A}$ which is rigid for every pair of R-modules M and N for which the weak λ-family is rigid.*

(b) *If the λ-family $\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ has the R-mono-splitting property (resp. is λ-R-free) then the new family $\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ has the R-mono-splitting property (resp. is λ-R-free).*

Proof: The statement (a) follows from [10, Proposition 1]. The statement (b) follows from the proof of [10, Proposition 1] and the arguments applied in the proof of [9, Proposition 2.3]. ∎

Throughout, we denote by ω the minimal countable ordinal number. We shall prove in Section 3 that there exists a rigid ω-family of R-free Kronecker modules with some extra properties. From this we shall derive the existence of large rigid families of Kronecker modules by applying the so-called Shelah elevator to move

ω-families up to any infinite cardinal λ. This is based on a result by Shelah [15], which can be formulated in terms of R_ω-modules.

By an R_ω-module we shall mean a system $\mathbf{F} = (F; F^{(k)})_{k\in\omega}$, where F is an R-module and $\{F^{(k)}: k \in \omega\}$ is a countable family of distinguished R-submodules of F. A morphism $\psi: \mathbf{F} \to \mathbf{G}$ between R_ω-modules $\mathbf{F}$ and $\mathbf{G} = (G; G^{(k)})_{k\in\omega}$ is an R-homomorphism $\psi: F \to G$ such that $\psi(F^{(k)}) \subseteq G^{(k)}$ for all $k \in \omega$.

Assume that λ is an infinite cardinal number. An R_ω-module $\mathbf{F}$ is said to be **λ-R-free** if the R-modules F, $F^{(k)}$, $F/F^{(k)}$ are free of rank λ for any $k \in \omega$.

A strong λ-family $\{\mathbf{F}_\beta, f_{\beta\gamma}, \mathbf{Q}_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ of R_ω-modules $\mathbf{F}_\beta$ and $\mathbf{Q}_i$ is said to be **λ-R-free** if the natural R-module embeddings $F_\beta^{(k)} \subseteq F_\beta$ and $Q_i^{(k)} \subseteq Q_i$ split and their complements are free R-modules of rank λ for all $\beta \subseteq \lambda$, $i \in \lambda$ and for all $k \in \omega$.

In [15] Shelah essentially has proved the important special case $R = \mathbb{Z}$ of the following non-trivial "Shelah elevator".

THEOREM 2.7: *For any infinite cardinal number λ there exists a strongly rigid λ-R-free strong λ-family $\{\mathbf{G}_\beta, g_{\beta\gamma}, \mathbf{Q}_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ of R_ω-modules.*

Proof: For the proof the reader is referred to [9] and [6, Section 3] (see also [15] for the proof in the case $R = \mathbb{Z}$). ∎

Definition 2.8: Let λ be an infinite cardinal number and let A be an R-algebra. Following [10], by a **λ-family of functors** from Mod(A) to $\mathcal{A}$ we shall mean a direct system

$$\mathbb{F} = \{\mathbb{F}_\beta, f_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$$

of R-linear additive functors $\mathbb{F}_\beta : \mathrm{Mod}(A) \longrightarrow \mathcal{A}$ connected by functorial morphisms $f_{\beta\gamma} : \mathbb{F}_\beta \to \mathbb{F}_\gamma$. The system $\mathbb{F}$ is said to be **rigid** if for any pair of modules M and N in Mod(A)

$$\mathrm{Hom}_\Lambda(\mathbb{F}_\beta(M), \mathbb{F}_\gamma(N)) = 0 \quad \text{if } \beta \not\subseteq \gamma,$$

and the natural R-homomorphism

$$\mathrm{Hom}_A(M, N) \xrightarrow{\simeq} \mathrm{Hom}_\Lambda(\mathbb{F}_\beta(M), \mathbb{F}_\gamma(N)), \tag{2.9}$$

$g \mapsto f_{\beta\gamma}(N) \circ \mathbb{F}_\beta(g)$, is an isomorphism for all $\beta \subseteq \gamma \subseteq \lambda$. We say that the system has the **R-mono-splitting property** if for any module M in Mod(A) and for all $\beta \subseteq \gamma \subseteq \lambda$ the R-homomorphism $f_{\beta\gamma}(M): \mathbb{F}_\beta(M) \to \mathbb{F}_\gamma(M)$ is an R-splittable monomorphism.

By applying Proposition 2.6, Theorem 2.7 and a method of Corner [4] the following important reduction result is proved in [10].

PROPOSITION 2.10: *Assume that R is a commutative ring with an identity element, $\mathcal{A}$ is a full subcategory of the category* $\mathrm{Mod}(\Lambda)$, *where Λ is an R-algebra, and $\mathcal{A}$ is closed under taking arbitrary direct sums and under taking extensions in* $\mathrm{Mod}(\Lambda)$.

(a) *If a weak ω-family $\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ in $\mathcal{A}$ is given, then for any infinite ordinal number λ there exists a strong λ-family*

$$\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$$

in $\mathcal{A}$, which is strongly rigid for every pair of R-modules M and N for which the weak ω-family $\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ is strongly rigid.

Moreover, if the given ω-family $\{U_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ has the R-mono-splitting property (resp. is ω-R-free), then the new family $\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ has the R-mono-splitting property (resp. is λ-R-free).

(b) *For any infinite cardinal number λ, for any R-algebra A generated by λ elements and for any strong λ-family $\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ in $\mathcal{A}$, there exists a λ-family $\mathbb{F} = \{\mathbb{F}_\beta, f_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ of R-linear additive functors $\mathbb{F}_\beta : \mathrm{Mod}(A) \longrightarrow \mathcal{A}$, which is rigid for every pair of A-modules M and N for which the strong λ-family $\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ is strongly rigid.*

Moreover, if the given λ-family $\{H_\beta, h_{\beta\gamma}, Q_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ has the R-mono-splitting property (resp. is λ-R-free), then the new family $\mathbb{F} = \{\mathbb{F}_\beta, f_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ has the R-mono-splitting property (resp. is λ-R-free).

3. Rigid families of Kronecker modules

Let R be a commutative ring with an identity element and let A be an R-algebra. We recall from Section 1 that

$$\Gamma_2(A) = \begin{pmatrix} A & A^2 \\ 0 & A \end{pmatrix}$$

is the Kronecker A-algebra and $\mathrm{Mod}_{pr}^{pr}(\Gamma_2(A))$ is the category of all A-projective Kronecker modules. Any such module P is identified with the system

$$P = (P', P'';\ \varphi', \varphi''), \tag{3.1}$$

where P' and P'' are projective A-modules and φ', φ'': $P' \to P''$ are A-homomorphisms.

We will frequently deal with a special kind of R-projective Kronecker module defined as follows.

Fix an infinite cardinal number λ. The Kronecker A-module P will be called **splittable free of rank** (λ, λ) if P' and P'' are free A-modules of rank λ, the A-homomorphisms $\varphi', \varphi'' : P' \to P''$ are splittable monomorphisms and the A-modules $P''/\operatorname{Im} \varphi'$ and $P''/\operatorname{Im} \varphi''$ are free. We denote by

$$\mathcal{SFK}r(A, \lambda) \tag{3.2}$$

the full subcategory of $\operatorname{Mod}_{pr}^{pr}(\Gamma_2(A))$ formed by splittable free Kronecker A-modules $P = (P', P''; \varphi', \varphi'')$ of rank (λ, λ) such that $P'' = \operatorname{Im} \varphi' + \operatorname{Im} \varphi''$.

The main result of this section is the following.

THEOREM 3.3: *Let R be a commutative ring, λ an infinite cardinal number and A an R-algebra which is generated by λ elements. Then there exists a direct system $\{H_\beta, h_{\beta\gamma}\}_{\beta \subseteq \gamma \subseteq \lambda}$ in the category $\operatorname{Mod}(\Gamma_2(A))$ of Kronecker A-modules with the following properties:*

(a) *Each H_β is in the category $\mathcal{SFK}r(A, \lambda)$ and each $h_{\beta\gamma}$ is an A-module homomorphism.*

(b) *The family $\{H_\beta, h_{\beta\gamma}\}_{\beta \subseteq \gamma \subseteq \lambda}$ is λ-A-free.*

(c) *For any pair of A-modules M and N we have*

$$\operatorname{Hom}_{\Gamma_2(R)}(M \otimes_A H_\beta, N \otimes_A H_\gamma) = \begin{cases} \operatorname{Hom}_A(M, N) & \text{if } \beta \subseteq \gamma, \\ 0 & \text{if } \beta \not\subseteq \gamma, \end{cases}$$

where the equality "=" means that the canonical R-homomorphism

$$\operatorname{Hom}_R(M, N) \longrightarrow \operatorname{Hom}_{\Gamma_2(R)}(M \otimes_A H_\beta, N \otimes_A H_\gamma),$$

$f \mapsto f \otimes h_{\beta\gamma}$, is bijective. In other words $\{H_\beta, h_{\beta\gamma}\}_{\beta \subseteq \gamma \subseteq \lambda}$ is a fully rigid system in the sense of Corner [5].

The proof is presented at the end of this section. We precede it by two preparatory important propositions.

By a slight modification of the proof of Proposition 2.3 in [9] we get the following result reducing the problem about the existence of rigid families to the existence of weak rigid families.

PROPOSITION 3.4: *If there exists a rigid weak λ-family of Kronecker R-modules in the category $\mathcal{SFK}r(R, \lambda)$ with the R-mono-splitting property (resp. λ-R-free), then there exists a strongly rigid strong λ-family of Kronecker R-modules in $\mathcal{SFK}r(R, \lambda)$ with the R-mono-splitting property (resp. λ-R-free).*

A crucial part of the proof of Theorems 3.3 and 1.2 is the following result which is a variation of a theorem of Baer (see Fuchs [7, Vol.2]) blended with an idea of Ringel (see the proof of Theorem 6.9 in [14]).

PROPOSITION 3.5: (a) *There exist a rigid weak ω-family $\{F_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ Kronecker R-module*

$$F_\beta = (F'_\beta, F''_\beta;\ \varphi'_\beta, \varphi''_\beta : F'_\beta \to F''_\beta)$$

of rank (ω,ω), and elements $e_\beta \in F''_\beta$ satisfying the following conditions:

(i) *the module F_β belongs to $\mathcal{SFK}r(R,\omega))$ for every $\beta \subseteq \omega$;*

(ii) *for all $\beta \subseteq \gamma \subseteq \omega$ the homomorphism $u_{\beta\gamma}$: $F_\beta \to F_\gamma$ is an R-splittable monomorphism and there is a Kronecker module isomorphism*

$$F_\gamma/\operatorname{Im} u_{\beta\gamma} \cong \bigoplus_{i\in\gamma\setminus\beta} \mathbf{R}(p_i(x)) \tag{3.5a}$$

where

$$\mathbf{R}(p_i(x)) = (R[x]/(p_i(x)), R[x]/(p_i(x)), \psi'_i, \psi''_i), \tag{3.5b}$$

ψ'_i is the identity map on $R[x]/(p_i(x))$ and $\psi''_i : R[x]/(p_i(x)) \to R[x]/(p_i(x))$ is induced by the scalar multiplication by x;

(iii) *$F''_\beta = \operatorname{Im}\varphi'_\beta + \operatorname{Im}\varphi''_\beta$, $\operatorname{Im}\varphi'_\beta \oplus Re_\beta = F''_\beta$, $\operatorname{Im}\varphi''_\beta \oplus Re_\beta = F''_\beta$ and $u_{\beta\gamma}(e_\beta) = e_\gamma$ for all $\beta \subseteq \gamma \subseteq \omega$.*

(b) *There exists a strong ω-family of splittable R-free Kronecker R-modules of rank (ω,ω) (that is, objects in $\mathcal{SFK}r(R,\omega)$), which is strongly rigid and ω-R-free.*

Proof: The statement (b) of the proposition follows from (a) and Proposition 3.4. The proof of (a) is divided into four steps.

STEP 1: We define a multiplicative subset $S \subset R[x]$ and $R[x]$-submodules L_β, $\beta \subseteq \omega$, of the localization $S^{-1}R[x]$ of $R[x]$ with respect to S with the following properties:

(a1) L_β is an S-torsion-free $R[x]$-submodule of rank 1 of $S^{-1}R[x]$ for any $\beta \subseteq \omega$.

(a2) If $\beta \subseteq \gamma \subseteq \omega$ then $L_\beta \subseteq L_\gamma$.

(a3) If N is an R-module and $\beta \cup \gamma \subseteq \omega$, then

$$\operatorname{Hom}_{R[x]}(L_\beta, N \otimes_R L_\gamma) = \begin{cases} N, & \text{if } \beta = \gamma, \\ 0, & \text{if } \beta\setminus\gamma \text{ is infinite,} \end{cases}$$

where $\mathrm{Hom}_{R[x]}(L_\beta, N\otimes_R L_\gamma) = N$ means that the R-homomorphism

$$\Psi_{N,L}\colon N \longrightarrow \mathrm{Hom}_{R[x]}(L_\beta, N\otimes_R L_\gamma),$$

$n \mapsto (\ell \mapsto n\otimes \ell)$, is bijective.

For this purpose we define inductively the countable set of polynomials

$$p_0, p_1, p_2, \dots p_n, \dots \in R[x]$$

by setting $p_0 = x+1$ and $p_{i+1} = 1 + x\cdot p_0 p_1 \cdots p_i$. Observe that

(i) $p_i(0) = 1$ for all $i \in \mathbb{N}$, and

(ii) the polynomials p_i, $i\in\mathbb{N}$, are pairwise comaximal, that is, for any $i \neq j$ there exist $q_i, q_j \in R[x]$ with $1 = p_i q_i + p_j q_j$.

We take for S the multiplicative closure in $R[x]$ of the set $\{p_0, p_1, p_2, \dots\}$. It follows that S has no zero divisors in $R[x]$, and $R[x]$ becomes a subring of the localization $S^{-1}R[x]$. As in [9], for any $\beta \subseteq \omega$ we define L_β to be the $R[x]$-submodule

$$L_\beta = \left\{ \frac{f}{p_{i_0}\cdot \dots \cdot p_{i_k}},\ f\in R[x], \text{all } i_0,\dots,i_k \text{ are distinct in } \beta \right\} \subseteq S^{-1}R[x]$$

of $S^{-1}R[x]$. The properties **(a1)** and **(a2)** easily follow from the definition.

Note also that if $i\in\beta$, then $p_i L_\beta = R[x] \oplus F'_{\beta\smallsetminus\{i\}}$, and if $i \in \omega\smallsetminus\beta$, then $p_i L_\beta = p_i R[x] \oplus F'_\beta$. It follows that in any case $p_i L_\beta$ has a free R-module complement in L_β and the module $N\otimes_R L_\beta$ is S-torsion-free for any R-module N. Then the property **(a3)** follows by applying the rank 1 considerations as in Baer's theorem in Fuchs [7, Vol.2, pp. 110 and 124] (see also [9, Lemma 3.1]).

STEP 2: For any $\beta\subseteq\omega$ we define an R-free Kronecker module

$$F_\beta = (F'_\beta, F''_\beta;\ \varphi'_\beta, \varphi''_\beta\colon F'_\beta \to F''_\beta) \tag{3.6}$$

in $\mathcal{SFK}r(R,\omega)$, where F'_β and F''_β are R-submodules of L_β defined as follows. Let d_i be the degree of p_i. We set

$$F_i^{(0)} = \sum_{j=0}^{d_i-1} R\frac{x^j}{p_i} = \bigoplus_{j=0}^{d_i-1} R\frac{x^j}{p_i} \subseteq S^{-1}R[x],$$

and if $\beta\subseteq\omega$, we define F'_β to be the R-submodule

$$F'_\beta = \sum_{i\in\beta} F_i^{(0)} = \bigoplus_{i\in\beta} F_i^{(0)}$$

of $S^{-1}R[x]$. Since $F_i^{(0)}$ is the p_i-primary component of F'_β and $p_i \neq p_j$ for $i \neq j$, then the primary decomposition theorem yields $F'_\beta = \bigoplus_{i\in\beta} F_i^{(0)}$ and $L_\beta = F'_\beta \oplus R[x]$ is an R-module decomposition by the partial fraction argument based on the property (ii) above (see [9, p. 35]).

Now we define F''_β to be the R-submodule

$$F''_\beta = F'_\beta + R1 \subseteq S^{-1}R[x]$$

of $S^{-1}R[x]$. Observe that $F'_\beta \cdot x \subseteq F''_\beta$, $F''_\beta = F'_\beta \oplus R1$ and $R1 = F''_\beta \cap R[x]$. Finally, we define two R-monomorphisms

$$\varphi'_\beta, \varphi''_\beta \colon F'_\beta \longrightarrow F''_\beta$$

where φ'_β is the natural embedding $F'_\beta \hookrightarrow F''_\beta$, and φ''_β is defined by the formula $\varphi''_\beta(f) = f \cdot x$, that is, φ''_β is the scalar multiplication by x.

The Kronecker R-modules $F_\beta = (F'_\beta, F''_\beta;\ \varphi'_\beta, \varphi''_\beta)$, $\beta \subseteq \omega$, (3.6) have the following properties:

(a4) If $\beta \subseteq \omega$ is infinite, then F_β is in $\mathcal{SFKr}(R,\omega)$, and there are R-module decompositions

$$F''_\beta = F'_\beta \oplus R1 \subseteq S^{-1}R[x], \quad F''_\beta = (\mathrm{Im}\ \varphi''_\beta) \oplus R1 \subseteq S^{-1}R[x],$$
$$\mathrm{Im}\ \varphi'_\beta + \mathrm{Im}\ \varphi''_\beta = F''_\beta. \tag{3.7}$$

(a5) For any $\beta, \gamma \subseteq \omega$ the sequence

$$0 \longrightarrow F_{\beta\cap\gamma} \longrightarrow F_\beta \oplus F_\gamma \longrightarrow F_{\beta\cup\gamma} \longrightarrow 0$$

is exact, where the maps are the natural ones induced by $u_{\beta\cap\gamma,\beta}$, $u_{\beta\cap\gamma,\gamma}$, $u_{\beta,\beta\cup\gamma}$ and $u_{\gamma,\beta\cup\gamma}$.

The property **(a5)** follows easily from the definition of the modules F_β and the maps $u_{\gamma,\beta}$.

Now we shall prove **(a4)**. Since the decomposition $F''_\beta = F'_\beta \oplus R1$ is obvious it remains to show that Im φ''_β has the R-free complement $1R$ in F''_β.

To see this we consider $g \in F_i^{(0)}x \cap 1R$. Then there exists a polynomial $f(x) \in R[x]$ such that $f(0) = 0$ and $g(x) = f(x)/p_i(x)$. Hence $f(x) = g(x)p_i(x)$. Since $0 = f(0) = g(0)p_i(0)$, $g \in R$ and $p_i(0) = 1$ by the choice of p_i then $g = g(0) = 0$. It follows Im φ''_β has a free complement $1R$ in F''_β. The R-ranks of the free modules can be computed easily. The remaining equality in (3.7) follows from the easy observation that $R1 \subseteq F'_\beta + F'_\beta x$.

STEP 3: We define the element $e_\beta \in F''_\beta$ to be the element $1 \in F''_\beta$. Since $\operatorname{Im}\varphi'_\beta = F'_\beta$, then (3.7) yields $\operatorname{Im}\varphi'_\beta \oplus Re_\beta = F''_\beta$ and $\operatorname{Im}\varphi''_\beta \oplus Re_\beta = F''_\beta$ for all $\beta \subseteq \omega$.

If $\beta \subseteq \gamma \subseteq \omega$, then the embedding $L_\beta \subseteq L_\gamma$ induces the R-module embeddings $u'_{\beta\gamma} : F'_\beta \subseteq F'_\gamma$ and $u''_{\beta\gamma} : F''_\beta \subseteq F''_\gamma$ such that

$$u_{\beta\gamma} = (u'_{\beta\gamma}, u''_{\beta\gamma}) : F_\beta \longrightarrow F_\gamma \tag{3.8}$$

is an embedding of Kronecker modules in $\mathcal{SFK}r(R,\omega)$.

It follows from the definition that $u_{\beta\gamma}(e_\beta) = e_\gamma$ for all $\beta \subseteq \gamma \subseteq \omega$. Hence we easily conclude that there is a Kronecker R-module isomorphism

$$F_\gamma / \operatorname{Im} u_{\beta\gamma} \cong \bigoplus_{i \in \gamma \setminus \beta} (\overline{F}'_{\gamma,i}, \overline{F}''_{\gamma,i}, \xi'_{\gamma,i}, \xi''_{\gamma,i}) \tag{3.8a}$$

where $\overline{F}'_{\gamma,i} = \overline{F}''_{\gamma,i} = F_i^{(0)}$, $\xi'_{\gamma,i}$ is the identity map and $\xi''_{\gamma,i} : F_i^{(0)} \to F_i^{(0)}$ is induced by the scalar multiplication by x. It is easy to see that for every i there is a Kronecker R-module isomorphism

$$(\overline{F}'_{\gamma,i}, \overline{F}''_{\gamma,i}, \xi'_{\gamma,i}, \xi''_{\gamma,i}) \cong \mathbf{R}(g_i(x)).$$

STEP 4: Now we shall construct a rigid weak ω-family $\{F_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ of Kronecker R-modules in the category $\mathcal{SFK}r(R,\omega)$.

For this purpose choose any infinite subset β_0 of ω such that $\omega \setminus \beta$ is also infinite. It follows from **(a4)** that the family F_β, $\beta_0 \subseteq \beta \subseteq \omega$, constructed in Step 2 is in $\mathcal{SFK}r(R,\omega)$. We take for $u_{\beta\gamma}$ the Kronecker R-module embeddings (3.8).

If we relabel $\omega\backslash\beta_0$ by ω, then we get an ω-family $\{F_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ of some of the original Kronecker R-modules F_β. We shall show that $\{F_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ has the required properties.

By **(a4)** and **(a5)**, $\{F_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ is a weak ω-family in $\mathcal{SFK}r(R,\omega)$.

In the proof that $\{F_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ has the properties **(W1)**–**(W3)** (see Section 2) we shall apply the following property:

(a6) If $f = (f', f'')$: $F_\beta \longrightarrow N \otimes_R F_\gamma$ is a homomorphism of Kronecker R-modules, then there exists an $R[x]$-homomorphism $\tilde{f}''$: $L_\beta \to N \otimes_R L_\gamma$ such that the diagram

$$\begin{array}{ccc} F''_\beta & \xrightarrow{u_\beta} & L_\beta \\ \Big\downarrow f'' & & \Big\downarrow \tilde{f}'' \\ N \otimes_R F''_\gamma & \xrightarrow{id_N \otimes u_\gamma} & N \otimes_R L_\gamma \end{array}$$

is commutative, where u_β is the natural embedding $F''_\beta = F'_\beta \oplus R1 \subseteq F'_\beta \oplus R[x] = L_\beta$.

In order to define $\tilde{f}''$ we recall that

$$f' \in \mathrm{Hom}_R(F'_\beta, N \otimes_R F'_\gamma), \quad f'' \in \mathrm{Hom}_R(F''_\beta, N \otimes_R F''_\gamma)$$

are such that

$$(*) \qquad (id_N \otimes \varphi'_\gamma) f' = f'' \varphi'_\beta \quad \text{and} \quad (id_N \otimes \varphi''_\gamma) f' = f'' \varphi''_\beta.$$

We shall identify the Kronecker R-module $N \otimes_R F_\gamma$ with the system

$$(N \otimes_R F'_\gamma, N \otimes_R F''_\gamma, id_N \otimes \varphi'_\gamma, id_N \otimes \varphi''_\gamma),$$

where $id_N \otimes \varphi'_\gamma$ is the natural embedding and $id_N \otimes \varphi''_\gamma$ is the scalar multiplication by x.

We define $\tilde{f}''$: $L_\beta \to N \otimes_R L_\gamma$ as follows. If ℓ is an element of L_β and $\ell = \ell' + h(x)$, $\ell' \in F'_\beta$, $h(x) \in R[x]$, we set

$$\tilde{f}'' = (id_N \otimes u_\gamma)(f''(\ell')) + [(id_N \otimes u_\gamma) f''(1_R)] \cdot h(x). \tag{3.9}$$

It is clear that $\tilde{f}''$ is an R-homomorphism and the diagram above is commutative.

We shall show that $\tilde{f}''$ is an $R[x]$-homomorphism. For this purpose we note that $\tilde{f}''(m) = (id_N \otimes u_\gamma) f''(m)$ for all $m \in F''_\beta$. Hence, in view of $(*)$, we get

$$\begin{aligned} f''(\ell' \cdot x) &= f'' \varphi''_\beta(\ell') \\ &= (id_N \otimes \varphi''_\gamma) f'(\ell') \\ &= [(id_N \otimes \varphi'_\gamma) f'(\ell')] \cdot x \\ &= [f'' \varphi'_\beta(\ell')] \cdot x \\ &= [f''(\ell')] \cdot x \end{aligned}$$

for any $\ell' \in F'_\beta$. It follows that

$$\begin{aligned} \tilde{f}''(\ell' \cdot x) &= (id_N \otimes u_\gamma) f''(\ell' \cdot x) \\ &= (id_N \otimes u_\gamma)(f''(\ell') \cdot x) \\ &= [(id_N \otimes u_\gamma) f''(\ell')] \cdot x \\ &= [\tilde{f}''(\ell')] \cdot x \end{aligned}$$

for any $\ell' \in F'_\beta$, and therefore $\tilde{f}''$ is an $R[x]$-homomorphism.

Now we shall prove that $\{F_\beta, u_{\beta\gamma}\}_{\beta \subseteq \gamma \subseteq \omega}$ has the properties **(W1)–(W3)** (see Section 2).

For **(W1)** we apply Remark 2.3 (b). It follows from the tensor product adjoint formula that in order to prove the condition **(W4)** for any pair of R-modules M and N, it is enough to prove it for $M = R$ and for any N.

Assume that $f = (f', f'')\colon F_\beta \longrightarrow N \otimes_R F_\gamma$ is a homomorphism of Kronecker R-modules. If $\beta \backslash \gamma$ is infinite, then by applying **(a3)** we get $\tilde{f''} = 0$; hence $f'' = 0$ and $f = 0$ as well. Then the bottom equality in **(W4)** follows.

In order to prove the top equality in **(W4)** assume that $\beta = \gamma$. It is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} N & \xrightarrow{\Psi_{N,F}} & \mathrm{Hom}_{\Gamma_2(R)}(F_\beta, N \otimes_R F_\gamma) \\ \downarrow{\scriptstyle id_N} & & \downarrow{\scriptstyle \Theta} \\ N & \xrightarrow{\Psi_{N,L}} & \mathrm{Hom}_{R[x]}(L_\beta, N \otimes_R L_\gamma) \end{array}$$

where $\Psi_{N,L}$ is defined in **(a3)**, $\Psi_{N,F}$ is defined analogously and, given a homomorphism $f = (f', f'') : F_\beta \to N \otimes_R F_\gamma$ of Kronecker R-modules, we take for $\Theta(f)$ the $R[x]$-homomorphism $\tilde{f''}$ defined by the formula (3.9).

It is clear that Θ is an injective R-homomorphism. Since, according to **(a3)**, the map $\Psi_{N,L}$ is bijective, then $\Psi_{N,F}$ is bijective and **(W4)** follows.

In order to prove **(W2)** assume that $g = (g', g'') : F_\beta \to \bigoplus_{\gamma \in \Gamma} N \otimes_R F_\gamma$ is a homomorphism of Kronecker R-modules, where Γ is an arbitrary set. It follows from **(a6)** that the R-homomorphism g'' extends to an $R[x]$-homomorphism $\tilde{g''}$ making the following diagram

$$\begin{array}{ccc} F''_\beta & \xrightarrow{u_\beta} & L_\beta \\ \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle \tilde{g''}} \\ \bigoplus_{\gamma\in\Gamma} N \otimes_R F''_\gamma & \xrightarrow{\bigoplus_{\gamma\in\Gamma} id_N \otimes u_\gamma} & \bigoplus_{\gamma\in\Gamma} N \otimes_R L_\gamma \end{array}$$

commutative. Since L_β is of a "rank-one type" $R[x]$-module, then the $R[x]$-homomorphism $\tilde{g''}$ factors through a finite direct sum $\bigoplus_{\gamma\in\Gamma_0} N \otimes_R L_\gamma$ containing the element $\tilde{f''}(1)$. It follows that g'' factors through a finite direct sum $\bigoplus_{\gamma\in\Gamma_0} N \otimes_R F''_\gamma$. Hence **(W2)** easily follows, because $\varphi'_\beta : F'_\beta \to F''_\beta$ is the natural embedding.

The property **(W3)** follows in a similar way by applying the definition of the Kronecker modules F_β and the choice of β_0.

Consequently, the ω-family $\{F_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\omega}$ of Kronecker R-modules in the category $\mathcal{SFK}r(R,\omega)$ is a rigid weak ω-family satisfying the required conditions (i)–(iii).

This finishes the proof of the statement (a) and of the proposition. ∎

Proof of Theorem 3.3: Let λ be an infinite cardinal number and let A be an R-algebra which is generated by the set $\{a_i\}_{i\in\lambda}$. It follows from Theorem 2.7 that there exists a strongly rigid λ-R-free strong λ-family of R_ω-modules.

Let $\{\mathbf{G}_\beta, g_{\beta\gamma}, \mathbf{Q}_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ be such a family and let

$$\mathbf{G}_\beta = \{G_\beta; G_\beta^{(k)}\}_{k\in\omega}$$

for $\beta \subseteq \lambda$. Following Brenner and Butler [3] (see also Corner [4, p. 162]) we shall modify the family in two steps as follows.

STEP 1: Since any $\mathbf{G}_\beta$ is λ-R-free, then we can choose a free basis $\{g_i, g_i', i\in\lambda\}$ for $G_\emptyset$. Define

$$A_i' = \{a\otimes g_i + a_i a\otimes g_i', a\in A\} \subseteq A\otimes_R G_\emptyset \subseteq A\otimes_R G_\beta$$

which is isomorphic to A_R. If $A^* = \sum_{i\in\lambda} A_i' \subseteq A\otimes_R G_\beta$, then $A^* = \bigoplus_{i\in\lambda} A_i' \cong \bigoplus_\lambda A$ has A-free complement $\bigoplus_{i\in\lambda} A\otimes g_i \cong \bigoplus_\lambda A$ in $A\otimes_R G_\emptyset$ (hence has A-free complement in $A\otimes_R G_\beta$). Moreover, we have (see [9, p. 39, Lemma 4.1])

$$(*)\qquad \{\Phi\in \mathrm{Hom}(A\otimes_R \mathbf{G}_\beta, A\otimes_R \mathbf{G}_\beta), \Phi(A^*)\subseteq A^*\} = A(id_A\otimes g_{\beta\beta}).$$

STEP 2: We shall modify the given rigid λ-family of R_ω-modules and obtain another one which we will also denote by $\{\mathbf{G}_\beta, g_{\beta\gamma}, \mathbf{Q}_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$.

First we shift the indices $k\in\omega$ by $k\mapsto k+2$ and we derive the R_ω-modules $\mathbf{G}_\beta = (G_\beta; G_\beta^{(k)})_{2\le k\in\omega}$ for $\beta\subseteq\lambda$. Recall from Section 2 that for any $\beta\subseteq\lambda$ there is an exact sequence

$$0 \longrightarrow \mathbf{G}_\emptyset \longrightarrow \mathbf{G}_\beta \longrightarrow \bigoplus_{i\in\beta} \mathbf{Q}_i \longrightarrow 0.$$

Next we add two distinguished λ-free submodules G_β^0, G_β^1 to $\mathbf{G}_\beta$ and Q_i^0, Q_i^1 to $\mathbf{Q}_i$, respectively, and the resulting new family $\{\mathbf{G}_\beta, g_{\beta\gamma}, \mathbf{Q}_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ will be the desired one. Let $G_\beta^0 = \bigoplus_{i\in\lambda} R(g_i + g_i')$, $G_\beta^1 = G_\beta$ and $Q_i^0 = 0$, $Q_i^1 = Q_i$. It is easy to check that the new family satisfies the required conditions.

STEP 3: Now we use the new λ-family $\{\mathbf{G}_\beta, g_{\beta\gamma}, \mathbf{Q}_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ of R_ω-modules and a given rigid ω-family of Kronecker R-modules (given in Proposition 3.5) to produce a λ-A-free direct system $\{H_\beta, h_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ of Kronecker A-modules in $\mathcal{SFK}r(A,\lambda)$.

It follows from Proposition 3.5 that there exists a strongly rigid ω-R-free ω-family of Kronecker R-modules in $\mathcal{SFK}r(R,\omega)$ satisfying the conditions in Proposition 3.5.

Let $\{F_\beta, f_{\beta\gamma}, Q'_i\}_{\beta\subseteq\gamma\subseteq\omega, i\in\omega}$ be such a family, where

$$F_\beta = (F'_\beta, F''_\beta;\ \varphi'_\beta, \varphi''_\beta\colon F'_\beta \to F''_\beta).$$

For $k\in\mathbb{N}$ we put $F_k = F_{\{k\}}$. Then $f_{\beta\gamma} = (f'_{\beta\gamma}, f''_{\beta\gamma})\colon F_\beta \to F_\gamma$ is an R-splittable monomorphism for all $\beta\subseteq\gamma\subseteq\omega$.

Let $\{\mathbf{G}_\beta, g_{\beta\gamma}, \mathbf{Q}_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ be the new λ-family above and

$$\mathbf{G}_\beta = (G_\beta, G_\beta^{(k)})_{k\in\omega}, \quad \text{where } G_\beta^{(k)} \subseteq G_\beta.$$

We can construct a new family in such a way that the modules $G_\beta^{(k)}\subseteq G_\beta$ have free basis elements chosen like in the proof of Proposition 3.4 of [9, p. 37].

Since $\mathbf{G}_\beta$ is λ-R-free, then the map $id_A\otimes v_{\beta,k}\colon A\otimes_R G_\beta^{(k)} \longrightarrow A\otimes_R G_\beta$ induced by the natural injection $v_{\beta,k}\colon G_\beta^{(k)} \hookrightarrow G_\beta$ is injective and A-splittable.

Moreover, the modules $A\otimes_R G_\beta^{(k)}$, $A\otimes_R G_\beta$ and the complement of $\mathrm{Im}(id_A\otimes v_{\beta,k})$ in $A\otimes_R G_\beta$ are A-free of rank λ.

Since $f_{k\omega} = (f'_{k\omega}, f''_{k\omega})\colon F_k \to F_\omega$ is an R-splittable monomorphism for any $k\in\omega$, then the map

$$id_A\otimes v_{\beta,k}\otimes f_{k\omega} : A\otimes_R G_\beta^{(k)}\otimes_R F_k \longrightarrow A\otimes_R G_\beta\otimes_R F_\omega$$

induced by the monomorphisms $v_{\beta,k} : G_\beta^{(k)} \hookrightarrow G_\beta$ and $f_{k\omega}$ is injective for any $k\in\omega$ and any $\beta\subseteq\lambda$.

We define the Kronecker A-module $A\otimes_R G_\lambda\otimes_R F_\omega$ to be the Kronecker R-module F_ω tensored with the free A-module $A\otimes_R G_\lambda$ of rank λ. For any $\beta\subseteq\lambda$ we set

$$H_\beta = \sum_{k\in\omega}\mathrm{Im}(id_A\otimes v_{\beta,k}\otimes_R f_{k\omega}) \ \subseteq\ A\otimes_R G_\beta\otimes_R F_\omega.$$

More explicitly, we have

$$H_\beta = (H'_\beta, H''_\beta;\ \psi'_\beta, \psi''_\beta : H'_\beta \to H''_\beta)$$

where

$$H'_\beta = \sum_{k\in\omega} \mathrm{Im}(id_A \otimes v_{\beta,k} \otimes_R f'_{k\omega}) \subseteq A \otimes_R G_\beta \otimes_R F'_\omega,$$
$$H''_\beta = \sum_{k\in\omega} \mathrm{Im}(id_A \otimes v_{\beta,k} \otimes_R f''_{k\omega}) \subseteq A \otimes_R G_\beta \otimes_R F''_\omega,$$

and the homomorphisms $\psi'_\beta, \psi''_\beta : H'_\beta \to H''_\beta$ are induced by the maps $\sum_k id_{A\otimes G_\beta^{(k)}} \otimes \varphi'_k$ and $\sum_k id_{A\otimes G_\beta^{(k)}} \otimes \varphi''_k$, respectively.

For any $\beta \subseteq \gamma \subseteq \lambda$ we define the map $h_{\beta\gamma}\colon H_\beta \to H_\gamma$ to be the restriction of the map $id_A \otimes g_{\beta\gamma} \otimes id_{F_\omega}\colon A \otimes_R G_\beta \otimes_R F_\omega \longrightarrow A \otimes_R G_\gamma \otimes_R F_\omega$ to the submodule H_β.

A straightforward analysis shows that H_β is a Kronecker A-module in $\mathcal{SFK}r(A, \lambda)$ and the family

$$\{H_\beta, h_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$$

is λ-A-free. For this purpose we apply the fact that the modules $\{F_\beta\}_{\beta\subseteq\omega}$ are in $\mathcal{SFK}r(R, \omega)$ and we have constructed the family $\{\mathbf{G}_\beta, g_{\beta\gamma}, \mathbf{Q}_i\}_{\beta\subseteq\gamma\subseteq\lambda, i\in\lambda}$ in such a way that the modules $G_\beta^{(k)} \subseteq G_\beta$ have free basis elements chosen like in the proof of Proposition 3.4 of [9, p. 37]. Then we can apply the arguments used in the proof of [9, Lemma 3.3 and Proposition 3.4].

It needs some mapping arguments to show that the direct system $\{H_\beta, h_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ has the properties (a)–(c) in Theorem 3.3. Fortunately, the proof given on pp. 40, 41 in [9] remains valid when we replace R_ϱ-modules $\mathbf{H}_\gamma$, $\gamma \subseteq \kappa$, in [9] by the Kronecker modules F_β, $\beta \subseteq \omega$. Here the equality (*) mentioned above applies. This finishes the proof of Theorem 3.3. ∎

Proof of Theorem 1.2: It follows from Theorem 3.3 that there exists a direct system $\{H_\beta, h_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ in the category $\mathrm{Mod}(\Gamma_2(R))$ of Kronecker R-modules satisfying the conditions (a)–(c) stated in Theorem 3.3.

We define a direct system $\mathbb{F} = \{\mathbb{F}_\beta, f_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$ of R-linear additive functors

$$\mathbb{F}_\beta\colon \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(\Gamma_2(R))$$

connected by injective functorial morphisms $f_{\beta\gamma} : \mathbb{F}_\beta \to \mathbb{F}_\gamma$ by setting

$$\mathbb{F}_\beta(M) = M \otimes_A H_\beta \quad \text{and} \quad f_{\beta\gamma}(M) = id_M \otimes h_{\beta\gamma}$$

for any right A-module M. Since $h_{\beta\gamma} : H_\beta \to H_\gamma$ is an A-splittable monomorphism, then the $\Gamma_2(R)$-homomorphism $f_{\beta\gamma}(M) : \mathbb{F}_\beta(M) \to \mathbb{F}_\gamma(M)$ is an R-splittable monomorphism. The remaining properties of the family $\mathbb{F}$ required in

Theorem 1.2 follow from the properties (a)–(c) stated in Theorem 3.3. The proof is complete. ∎

4. Open problems

In connection with the solution of the endomorphism ring problem given in Corollary 1.3, the following problem arises.

PROBLEM 4.1: *Prove that if K is a commutative field, then every K-algebra is isomorphic to an endomorphism algebra* End X *of a torsion-free module X over the polynomial K-algebra $K[t]$.*

In connection with representation embedding problems studied in [17] the following open problems are stated in [18].

PROBLEM $(\hat{\gamma}_1)$: *Find finite dimensional K-algebras Λ (resp. bipartite finite dimensional K-algebras S of the form*

$$S = \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}$$

where ${}_AM_B$ is an A-B-bimodule) for which the category $\mathrm{mod}(\Lambda)$ *(resp. the category* $\mathrm{prin}(R)^A_B$ *of prinjective modules* [16, Section 17.9]) *is of infinite representation type if and only if there exists a full faithful exact functor*

$$T\colon \mathrm{mod}(\Gamma) \longrightarrow \mathrm{mod}(\Lambda) \quad \textit{resp.} \quad T\colon \mathrm{mod}(\Gamma) \longrightarrow \mathrm{prin}(R)^A_B),$$

where Γ is a finite dimensional K-algebra of the form

$$\Lambda_N = \begin{pmatrix} F & {}_FN_G \\ 0 & G \end{pmatrix}, \tag{4.2}$$

F, G are division K-algebras and ${}_FN_G$ is an F-G-bimodule such that

$$(\dim {}_FN) \cdot (\dim N_G) = 4.$$

PROBLEM $(\hat{\gamma}_1^{sp})$: *Find finite dimensional K-algebras Λ (resp. bipartite algebras S of the form $S = \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}$) for which the category* $\mathrm{mod}(\Lambda)$ *(resp.* $\mathrm{prin}(S)^A_B$*) is of infinite representation type if and only if there exists a representation embedding functor* $\mathrm{mod}_{\mathrm{sp}}(\Gamma) \longrightarrow \mathrm{mod}(\Lambda)$ *(resp.* $\mathrm{mod}_{\mathrm{sp}}(\Gamma) \longrightarrow \mathrm{prin}(S)^A_B$*), where Γ is a finite dimensional K-algebra of the form Λ_N as in Problem $(\hat{\gamma}_1)$, and* $\mathrm{mod}_{\mathrm{sp}}(\Gamma)$ *is the full subcategory of* $\mathrm{mod}(\Gamma)$ *consisting of modules having the socle projective.*

Note that the characterization of minimal representation-infinite loop-finite artin algebras given by Skowroński in [21, Theorem 4.1] yields

THEOREM 4.3: *Suppose that K is a field and Λ is a finite dimensional K-algebra which is loop-finite in the sense of [20], that is, the infinite radical* $\mathrm{rad}^{\infty}(X,X)$ *is zero for any indecomposable module X in* $\mathrm{mod}(\Lambda)$. *The Λ is of infinite representation type if and only if there exist division K-algebras F and G, an F-G-bimodule ${}_FN_G$ such that* $(\dim {}_FN)\cdot(\dim N_G) = 4$ *and a full faithful exact functor* $\mathrm{mod}(\Lambda_N) \longrightarrow \mathrm{mod}(\Lambda)$, *where Λ_N is a finite dimensional K-algebra of the form* (4.2)

The proof of Theorem 4.3 essentially depends on the results of Skowroński in [21, Theorem 4.1] and in [22, pp. 651–652] (see [12]).

In connection with Theorem 4.3 and the main result of the present paper, the following problem arises.

PROBLEM 4.4: *Prove that if K is a commutative field, R is a commutative K-algebra and Λ is a representation-infinite loop-finite and finite dimensional K-algebra, then for any R-algebra A generated by at most λ elements, where λ is an infinite cardinal number, there exists a direct system*

$$\mathbb{F} = \{\mathbb{F}_\beta, u_{\beta\gamma}\}_{\beta\subseteq\gamma\subseteq\lambda}$$

of R-linear additive functors $\mathbb{F}_\beta\colon \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(R\otimes_K \Lambda)$ *connected by functorial morphisms* $u_{\beta\gamma}\colon \mathbb{F}_\beta \to \mathbb{F}_\gamma$ *satisfying the conditions analogous to* (i)–(iii) *stated in Theorem* 1.2.

In view of Theorem 4.3, Problem 4.4 reduces to algebras Λ of the form Λ_N (4.1), where F, G are finite dimensional division K-algebras and ${}_FN_G$ is a finite dimensional F-G-bimodule such that $(\dim {}_FN)\cdot(\dim N_G) = 4$.

COROLLARY 4.5: *If K is an algebraically closed field, then Problem* 4.4 *has a positive solution.*

Proof: By the remark above, without loss of generality we can suppose that $\Lambda = \Lambda_N$, where ${}_FN_G$ is a bimodule satisfying the conditions above. Since K is algebraically closed, then $F \cong G \cong K$ and there is a bimodule isomorphism ${}_FN_G \cong {}_KK_K$ along the ring isomorphisms $F \cong K$ and $G \cong K$. Consequently, the algebra Λ_N is isomorphic to the Kronecker K-algebra $\Gamma_2(K)$ (see (1.1)) and therefore there are ring isomorphisms $R\otimes_K\Lambda \cong R\otimes_K\Lambda_N \cong \Gamma_2(R)$. Then, according to Theorem 1.2 the corollary follows. ∎

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